UNIVERSITY OF LONDON

GENERAL CERTIFICATE OF EDUCATION EXAMINATION

Advanced Level

SUMMER, 1960

PURE MATHEMATICS.—I

MONDAY, June 20.—*Morning*, 9.30 *to* 12.30

Not more than EIGHT *questions are to be attempted.*

1. Express

$$\frac{3x-1}{x^3-2x^2+x-2}$$

in the form

$$\frac{Ax}{x^2+1}+\frac{B}{x^2+1}+\frac{C}{x-2}.$$

Hence, if $-1 < x < 1$, find in the expansion of the expression in ascending powers of x

(*a*) the coefficient of x^6,

(*b*) the coefficient of x^n when (i) n is even, (ii) n is odd.

2. (i) If α and β are the roots of the equation $x^2-3x-7=0$, form the equation whose roots are

$$\alpha+\frac{1}{\beta} \quad \text{and} \quad \beta+\frac{1}{\alpha}.$$

(ii) Jones wishes to invest a sum of money so that he will receive £100 for each year from the age of 65 years to that of 74 years inclusive. Compound interest is reckoned at 4 per cent per annum. If Jones is 40 years old when he makes the investment, how much should it be ?

3. A woman buys 3 raffle tickets, 2 coloured red and 1 green. There are 3 prizes for the red tickets of which 81 are sold. There are 4 prizes for the green tickets of which 96 are sold. What is the probability that she will win (*a*) three prizes, (*b*) two prizes ?

4. (i) By putting $x - \frac{2}{x} = y$, solve the equation

$$x^4 - 3x^3 - 2x^2 + 6x + 4 = 0.$$

(ii) Find the range of values of $\frac{x^2}{x^2 - 1}$ for which x is real.

Sketch the graph of the curve $y = \frac{x^2}{x^2 - 1}$.

5. (i) Prove that

$$\sin 2A \sec (n+1) A \sec (n-1) A = \tan (n+1) A - \tan (n-1)A.$$

Use this result to show that the sum of the series

$$\sec A \sec 3A + \sec 3A \sec 5A + \ldots + \sec (2r-1) A \sec (2r+1) A$$

is $$\{\tan (2r+1) A - \tan A\} \operatorname{cosec} 2A.$$

(ii) Solve the equation

$$2 \sin \theta - 3 \cos \theta = 1$$

for angles lying between 0° and 360°.

6. A line of slope, in an easterly direction, of a plane hillside is inclined at an angle α to the horizontal. A line of slope of the hillside in a southerly direction is inclined at an angle β to the horizontal. Prove that the actual inclination, θ, of the hillside to the horizontal is

$$\theta = \tan^{-1} (\tan^2 \alpha + \tan^2 \beta)^{\frac{1}{2}}.$$

A vertical pole of height h is placed on top of the hill. Show that the angle ϕ subtended by it at a point distant a down the line of greatest slope through the foot of the pole is given by

$$\tan \phi = \frac{h}{a \sec \theta + h \tan \theta}.$$

Find ϕ, if $h = 16$, $a = 36$, $\alpha = 30°$ and $\beta = 45°$.

7. Draw the graph of $y = \sin\theta + \sin 2\theta$ for values of θ between 0 and 2π radians.

Hence solve the equation $\theta = \sin\theta + \sin 2\theta$ for values of θ between 0 and 2π radians.

Determine also from the graph the range of values of k for which $\sin\theta + \sin 2\theta = k\theta$ has real roots in the interval $0 \leqslant \theta \leqslant 2\pi$.

8. A triangle is formed by the three lines

$$x + y = 1\ , \quad 3x - y = 7 \quad \text{and} \quad 3y = x + 3.$$

Calculate (*a*) the area of the triangle,

(*b*) the angles of the triangle,

(*c*) the coordinates of the circumcentre of the triangle.

9. (i) Differentiate $\log_e(k \sec x) + a^x$ with respect to x, a and k being constants.

(ii) If $x = \sin t$ and $y = \cos 2t$, prove that

$$\frac{d^2y}{dx^2} + 4 = 0\ .$$

(iii) Find the maximum and minimum values of

$$\frac{x - 3}{x^2 - x - 2}\ ,$$

and distinguish between them.

10. Find the area of the loop of the curve $y^2 = x(4 - x)^2$.

Also find the volume obtained by revolving the upper half of the loop through 4 right angles

(i) about the x-axis,

(ii) about the y-axis.