UNIVERSITY OF LONDON

GENERAL CERTIFICATE OF EDUCATION EXAMINATION

Advanced Level

SUMMER, 1951

PURE MATHEMATICS.—II

Examiners :
Professor R. L. GOODSTEIN, M.SC., PH.D., D.LIT.
E. E. IRONMONGER, Esq., B.SC.
E. S. MARTIN, Esq., PH.D.
H. E. PARR, Esq., M.A.

FRIDAY, June 15.—*Morning*, 9.30 *to* 12.30

[*Not more than* EIGHT *questions are to be attempted.*]

1. (i) By considering the roots of the equation

$$\sin 2\theta = \cos 3\theta,$$

prove that $\sin 18^\circ = (\sqrt{5}-1)/4$, and deduce the value of $\cos 36^\circ$.

(ii) Prove that

$$\cos^2\left(\frac{\pi}{8}-\alpha\right)-\cos^2\left(\frac{\pi}{8}+\alpha\right)=\frac{1}{\sqrt{2}}\sin 2\alpha.$$

2. The altitudes from the vertices A, B, C of a triangle are of lengths h_1, h_2, h_3, and the radius of the circumcircle is R. Prove that

(i) $h_1 \sin A = h_2 \sin B = h_3 \sin C$;

(ii) $h_1 \cos A + h_2 \cos B + h_3 \cos C = (a^2+b^2+c^2)/4R$.

3. A triangle ABC is drawn on a plane which makes an angle θ with the horizontal; A and B are on the same level, and C is below them; CA, CB make angles α, β with the horizontal plane. Prove that $\sin\theta = c\sin\beta/b\sin C$, and deduce that

$$\sin^2\theta\sin^2 C = \sin^2\alpha + \sin^2\beta - 2\sin\alpha\sin\beta\cos C.$$

4. Draw the graph of $y = \sin x^\circ$ for values of x from 0 to 180.

P is a point on a semicircle of which AB is the diameter. If the length of the arc AP is twice the distance of P from AB, find, from the graph or otherwise, the angle which AP subtends at the centre of the circle.

5. A square $ABCD$ has its opposite vertices A and C at $(2, -5)$ and $(-4, 3)$. Find the co-ordinates of B and D, and the equations of the lines which contain the sides of the square.

6. Show that the co-ordinates of any point lying on the straight line which passes through the point (a, b) and makes an angle α with the x-axis can be written in the form

$$x = a + r\cos\alpha, \qquad y = b + r\sin\alpha.$$

A line which makes an acute angle θ with the positive x-axis is drawn through the point P, whose co-ordinates are $(3, 4)$, to cut the curve $y^2 = 4x$ at Q and R. Show that the lengths of the segments PQ and PR are the numerical values of the roots of the equation

$$r^2\sin^2\theta + 4r(2\sin\theta - \cos\theta) + 4 = 0.$$

Hence find the gradients of the tangents from P to the curve.

7. P is any point on the curve $y^2 = 4ax$, and O is the origin; Q is the foot of the perpendicular from P to the y-axis, R is the foot of the perpendicular from Q to OP, and QR produced meets the x-axis at K. Prove that K is a fixed point, and find its co-ordinates.

Prove also that the locus of R is a circle.

8. (i) If $y=\left(\frac{x^2-2}{x^2+1}\right)^n$, prove that

$$\frac{dy}{dx}=\frac{6nxy}{(x^2-2)(x^2+1)}.$$

(ii) Explain, with reference to a diagram, the meaning of the statement " δy is approximately equal to $\frac{dy}{dx}\,\delta x$," where y is a function of x, and δx, δy are corresponding small increments in x, y.

Use this result to estimate the error made in calculating the area of a triangle ABC in which the sides a and b are measured accurately as 16 in. and 25 in., while the angle C is measured as 60° but is $\frac{1}{2}$° in error. (Give the answer in terms of π.)

9. Find (i) $\int\left(x+\frac{1}{x}\right)^2 dx$;

(ii) $\int_{\pi/6}^{\pi/2} \sin x \sin 2x\, dx$;

(iii) $\int_0^{\pi/2} \cos^3 x\, dx$.

10. (i) If $\frac{dy}{dx}=6x+2$, and $y=5$ when $x=0$, find y in terms of x.

(ii) Sketch the graph of $y^2=x(x-3)^2$, and find the area of the loop.